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Uniform Approximation by Quasi-Convex and Convex Functions*

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Given a bounded function f defined on a convex subset of R^n, the two problems considered are to find a quasi-convex (convex) function which is a best approximation to f under the uniform norm. It is shown that if $\bar{f}$ is the greatest quasi-convex (convex) minorant of f, then $f' = \bar{f} + c$, for some $c \geqq 0$, is the maximal best quasi-convex (convex) approximation to f. Furthermore, the nonlinear operator T defined by $T(f) = f'$ is a Lipschitzian selection operator with some least constant $C(T)$, where $C(T) \leq C(T')$ for all Lipschitzian operators T' which map f to one of its best approximations. Thus T is optimal in this sense.

1. INTRODUCTION

In this article are considered two problems of uniform approximation of a given function f by quasi-convex and convex functions. All functions are defined and bounded on a convex subset of R^n and no further conditions are imposed on this subset. It is shown that if $\bar{f}$ is the greatest quasi-convex (convex) minorant of f, then $f' = \bar{f} + c$, where $c = (1/2)\|f - \bar{f}\|$, is the maximal best quasi-convex (convex) approximation to f in the two respective problems. An explicit expression for the greatest quasi-convex minorant is derived in terms of f. It is shown that the mapping T with $T(f) = f'$ is Lipschitzian with a least constant $C(T)$, where $C(T)$ is the smallest among all such Lipschitzian mappings which map f to one of its best approximations; T is thus an optimal Lipschitzian selection operator.

Let $S \subset R^n$ be a nonempty convex set and let B denote the Banach space of all bounded real functions f on S with the uniform norm

$$\|f\| = \sup\{|f(s)| : s \in S\}.$$

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An f in B is said to be quasi-convex if

$$f(\lambda s+(1-\lambda)\,t) \leqq \max\{f(s), f(t)\} \tag{1.1}$$

for all $s, t \in S$ and all $0 \leqq \lambda \leqq 1$ [4, 5]. Similarly, f is convex if

$$f(\lambda s+(1-\lambda)\,t) \leqq \lambda f(s)+(1-\lambda) f(t) \tag{1.2}$$

for all s, $t \in S$ and all $0 \leqslant \lambda \leqslant 1$ [5, 6]. It can be shown that f is quasi-convex if and only if one of the following holds. (i) $f^{-1}(-\infty, \alpha]$ is convex for all real α, (ii) $f^{-1}(-\infty, \alpha)$ is convex for all real α. Let $K \subset B$ be the set of all quasi-convex (convex) functions for the two problems. Given f in B, let $\Delta(f)$ be the infimum of $\|f-k\|$ for k in K. The problem is to find an f' in K, called a best approximation to f from K, so that

$$\Delta(f)=\|f-f'\|=\inf\{\|f-k\|: k \in K\}.$$

We observe that the set of all convex functions is a closed convex cone; however, the set of all quasi-convex functions is a closed cone but not convex.

In general, the set K_f of all best approximations to f is not singleton. A Lipschitzian selection operator T is a nonlinear operator which maps each f in B to an f' in K_f and satisfies, for some least number $C(T)$,

$$\|T(f)-T(h)\| \leqq C(T)\,\|f-h\| \qquad \text{for all } f, h \in B.$$

T is an optimal Lipschitzian selection operator (OLSO) if $C(T) \leqq C(T')$ for all Lipschitzian selection operators T'. In this article, we obtain a best approximation f' to f so that T with $T(f)=f'$ is an OLSO. We determine $C(T)$ and the smallest number D so that

$$|\Delta(f)-\Delta(h)| \leqq D\,\|f-h\| \qquad \text{for all } f, h \in B. \tag{1.3}$$

Given an f in B, we define its greatest quasi-convex (convex) minorant $\bar{f}$ to be the largest quasi-convex (convex) function which does not exceed f at any point in S, viz.,

$$\bar{f}(s)=\sup\{k(s): k \in K, k(s) \leqq f(s) \text{ for all } s \text{ in } S\}, \qquad s \in S.$$

By (1.1) and (1.2) it is easy to verify that $\bar{f}$ is indeed quasi-convex (convex). We show that $\Delta(f)=(1/2)\,\|f-\bar{f}\|$ and $f'=\bar{f}+\Delta(f)$ is the maximal best approximation to f, i.e., $f' \geqq g$ for all best approximations g to f. Furthermore, T defined by $T(f)=f'$ is an OLSO with $C(T)=2$ and $D=1$. These results are established in Sections 3 and 4 for the two problems respectively. In Section 3, we also obtain an explicit expression for the greatest quasi-convex minorant $\bar{f}$ of a given f. In Section 2, we consider a uniform approximation problem in a general setting which includes the two problems as special cases and obtain results which lead to the derivation of f' and an OLSO T. Given f in B, let

$$L_f = \{k \in K: k(s) \leqq f(s) \text{ for all } s \text{ in } S\}.$$

Note that L_f is convex if K is the set of all convex functions. Let $\bar{\Delta}(f)$ be the infimum of $\|f-k\|$ for k in L_f and consider the problem of finding a best approximation f' to f from L_f so that $\bar{\Delta}(f) = \|f-f'\|$. In Section 5, we observe that $\bar{f}$ is the maximal best approximation to f from L_f and T with $T(f) = \bar{f}$ is an OLSO with $C(T) = 1$ and $\bar{D} = 2$, where $\bar{D}$ is the smallest number satisfying (1.3) with $D = \bar{D}$ and $\Delta = \bar{\Delta}$.

The problems of quasi-convex and convex approximation on a real interval $I = [a, b]$ have been investigated earlier in [8, 10]. The main thrust there was algorithms for obtaining best approximations. An analysis of OLSOs for these problems on I and generalized isotone optimization on a partially ordered set appeared in [9, 11]. In this article, the quasi-convex and convex problems are considered in a more general setting of a convex subset S of R^n without any additional constraints on S. Consequently, the methods of analysis are different. Particularly, for the convex problem, a Hausdorff metric-like function d is defined on the subsets of $S \subset R$ with the following property. The mapping of epigraphs of functions in B to its convex hulls is nonexpansive with respect to d. This function plays a key role in the analysis of OLSOs. The quasi-convex problem considered earlier [10] on an interval I, could be expressed in a setting of isotone functions on totally ordered subsets of I by decomposing the cone of quasi-convex functions into convex cones of isotone functions. It was then possible to isolate a "function interval" $[u, v]$ of quasi-convex functions u, v so that u and v as well as any quasi-convex function in this interval are also best approximations. Due to a more general setting of the problem in this article such an approach is not possible; however, it has been possible to obtain an explicit expression for the greatest quasi-convex minorant in terms of f and then for the maximal best approximation f'. Applications of these problems have been indicated to curve fitting and estimation in [9]. If T is an OLSO, then $T(f)$ is a best fit to data points f, and $T(f)$ is least sensitive, among all best fits, to perturbations of f. Finding continuous selections, which are conceptually similar to OLSOs, has been a problem of considerable interest in the literature; for a survey see [1]. Finally, we point out that a class of approximation problems on the space of continuous functions including the quasi-convex and convex problems is considered in [12].

2. Preliminaries

In this section, we derive results which will be used later.

Let $G \subset B$ be nonempty. Given an f in B, let $\Delta(f)$ be the infimum of $\|f-g\|$ for g in G. We consider the problem of finding a best approximation f' in G so that $\Delta(f) = \|f-f'\|$.

We impose the following two conditions on G.

(i) If $g \in G$ then $g+c \in G$ for all real c.

(ii) If $G' \subset G$ is a set of functions uniformly bounded above on S, then the function g', which is the pointwise supremum of functions in G', is in G.

We observe that if $f, h \in B$ then

$$|\Delta(f) - \Delta(h)| \leqq \|f-h\|. \tag{2.1}$$

See, e.g., [3, p. 17]. This follows immediately from

$$\|h-g\| \leqq \|f-g\| + \|f-h\|$$

by taking the infimum over g in G and then interchanging f and h.

Proposition 2.1. *Let $f \in B$ and assume that conditions* (i) *and* (ii) *hold for G. Define*

$$G' = \{g \in G\colon g(s) \leqq f(s) \text{ for all } s \in S\}.$$

Then G' is nonempty. Let

$$\bar{f}(s) = \sup\{g(s)\colon g \in G'\}.$$

Then $\bar{f} \in G$,

$$\Delta(f) = (1/2)\|f-\bar{f}\|, \tag{2.2}$$

and $f' = \bar{f} + \Delta(f)$ is the maximal best approximation to f, i.e., $f' \geqq g$ holds for all best approximations g. Furthermore, if $f, h \in B$ then

$$\|f'-h'\| \leqq \|\bar{f}-\bar{h}\| + |\Delta(f) - \Delta(h)| \tag{2.3}$$

and

$$\|f'-h'\| \leqq \|\bar{f}-\bar{h}\| + \|f-h\|. \tag{2.4}$$

Proof. For g in G, let $\delta(g) = \|f-g\|$. If $g \in G$, then $g - \delta(g) \leqq f$. By condition (i), $g-\delta(g)$ is in G and hence in G'. Thus G' is nonempty and by condition (ii), $\bar{f} \in G$. Clearly, $g - \delta(g) \leqq \bar{f} \leqq f$. Consequently,

$$0 \leqq f - \bar{f} \leqq f - g + \delta(g) \leqq \|f-g\| + \delta(g) = 2\delta(g).$$

Hence $\delta(g) \geqq \delta(\bar{f})/2$ for all $g \in G$. Thus we have $\Delta(f) \geqq \delta(\bar{f})/2 = c$, say. Let $f' = \bar{f} + c$. By condition (i), $f' \in G$. Now $f - f' = f - \bar{f} - c$. Hence we have

$$-c \leqq f - f' \leqq \|f - \bar{f}\| - c = c.$$

It follows that (2.2) holds and f' is a best approximation. Now, if g is a best approximation, then $g - \Delta(f) \leqq f$. Since $g \in G$, by condition (i), $g - \Delta(f) \in G'$. Hence, $g - \Delta(f) \leqq \bar{f}$ or $g \leqq f'$. Inequality (2.3) is immediate and (2.4) follows from (2.1). The proof is now complete.

Using the definitions (1.1) and (1.2), one may easily verify that the above conditions (i) and (ii) hold with $G = K$ for the quasi-convex and convex problems. Hence Proposition 2.1 applies. These investigations are pursued further in the next two sections.

3. Approximation by Quasi-Convex Functions

In this section, we apply the results of Section 2 to the quasi-convex problem. We first derive an explicit expression for the greeatest quasi-convex minorant (GQM) $\bar{f}$ of f.

THEOREM 3.1. *Let Π be the set of all nonempty convex subsets of S. Let $f \in B$ and define for each $s \in S$ and $P \in \Pi$,*

$$\begin{aligned} f^0(s, P) &= \inf\{f(t): t \in S - P\}, && \text{if } s \notin P, \\ &= -\infty, && \text{otherwise.} \end{aligned}$$

Let

$$\bar{f}(s) = \sup\{f^0(s, P): P \in \Pi\}, \qquad s \in S.$$

Then $\bar{f} \in B$ and is the greatest quasi-convex minorant of f. Furthermore, for all real α, we have

$$\{s \in S: \bar{f}(s) < \alpha\} = \operatorname{co}\{s \in S: f(s) < \alpha\},$$

where $\operatorname{co}(A)$ is the convex hull of $A \subset R^n$, i.e., the smallest convex set containing A. Conversely, if the above equality holds for all α for some $\bar{f}$ in B, then $\bar{f}$ is the greatest quasi-convex minorant of f.

Proof. We first show that $\bar{f}$ is quasi-convex. Let $u, v \in S$, $0 < \lambda < 1$, and $x = \lambda u + (1 - \lambda) v$. We assert that

$$f^0(x, P) = \max\{f^0(u, P), f^0(v, P)\}, \tag{3.1}$$

for all P in Π. To show (3.1), let $P \in \Pi$ and assume that the left side of (3.1) is finite. Then $x \notin P$. Since P is convex this implies that at least one of u and v is not in P. Suppose that $u \notin P$ and $v \in P$. Then $f^0(x, P) = f^0(u, P)$, $f^0(v, P) = -\infty$. Hence (3.1) holds. The other two cases where $u \in P$, $v \notin P$ and $u \notin P$, $v \notin P$ are similar. This establishes (3.1). It follows at once that $\bar{f}(x) \leqq \max\{\bar{f}(u), \bar{f}(v)\}$, i.e., $\bar{f}$ is quasi-convex. Since $f^0(s, P) \leqq f(s)$ for all P, we have $\bar{f}(s) \leqq f(s)$. Again, since $f^0(s, P) \geqq -\|f\|$ if s is in $S-P$, we conclude that $\bar{f} \in B$.

To show that $\bar{f}$ is the GQM of f, let h in B be any quasi-convex function with $h \leqq f$. Let $s \in S$ and

$$Q = \{t \in S: h(t) < h(s)\}.$$

Then, by the quasi-convexity of h, Q is convex and $s \notin Q$. Since $h^0(s, P) \leqq h(s)$ for all P and $s \notin Q$, we conclude that $h^0(s, Q) = h(s)$. Hence $\bar{h}(s) = h(s)$. Again, since $h \leqq f$ we have that $h^0(s, P) \leqq f^0(s, P)$ for all P. Hence $\bar{h}(s) = h(s) \leqq \bar{f}(s)$. Thus $\bar{f}$ is the GQM.

To prove the remaining assertions, let $A = (-\infty, \alpha)$. Then, since $\bar{f} \leqq f$, we have $\bar{f}^{-1}(A) \supset f^{-1}(A)$. By convexity of $\bar{f}^{-1}(A)$, we have $\bar{f}^{-1}(A) \supset \operatorname{co}(f^{-1}(A)) = P$, say. Now let $s \in S$ and $\bar{f}(s) < \alpha$. We show that $s \in P$. Since P is convex, we have $f^0(s, P) \leqq \bar{f}(s) < \alpha$. Hence, by the definition of $f^0(s, P)$, either $s \in P$ or $s \in S-P$ and there exists $t \in S-P$ such that $f(t) < \alpha$ which implies the contradiction that $t \in P$. Hence $s \in P$. To show the converse, we note that $\bar{f}$ satisfying the given equality for all α is quasi-convex, because the right hand side of the equality defines a convex set. Now let h be quasi-convex and $h \leqq f$. Then we have $h^{-1}(A) \supset f^{-1}(A)$. Since the former set is convex, $h^{-1}(A) \supset \operatorname{co}(f^{-1}(A)) = \bar{f}^{-1}(A)$ holds for all α. This implies that $h \leqq \bar{f}$. The proof is complete.

We remark that the last statement of the above theorem immediately leads to an algorithm for computing $\bar{f}$ when S is a finite set.

THEOREM 3.2. *Let $f \in B$ and $\bar{f}$ be the greatest quasi-convex minorant of f. Then $\Delta(f) = (1/2)\|f - \bar{f}\|$, and $f' + \Delta(f)$ is the maximal best quasi-convex approximation to f. Furthermore, if $f, h \in B$, then*

$$\|\bar{f} - \bar{h}\| \leqq \|f - h\|. \tag{3.2}$$

Hence,

$$\|f' - h'\| \leqq \|f - h\|, \qquad \text{if} \quad \Delta(f) = \Delta(h), \tag{3.3}$$

and

$$\|f' - h'\| \leqq 2\,\|f - h\|. \tag{3.4}$$

The operator $T: B \to K$ defined by $T(f) = f'$ is an optimal Lipschitzian selection operator with $C(T) = 2$. Also $D = 1$.

Proof. The assertions concerning $\Delta(f)$ and f' follow from Proposition 2.1. We now show (3.2). Let $s \in S$. Given $\varepsilon > 0$, there exists $P \in \Pi$ with $s \notin P$ such that $\bar{f}(s) \leqq f^0(s, P) + \varepsilon/2$ and $u \in S - P$ such that $h^0(s, P) \geqslant h(u) - \varepsilon/2$. Then $\bar{h}(s) \geqq h^0(s, P)$ and $f^0(s, P) \leqq f(u)$. Hence,

$$\bar{f}(s) - \bar{h}(s) \leqq f^0(s, P) - h^0(s, P) + \varepsilon/2 \leqq f(u) - h(u) + \varepsilon \leqq \|f - h\| + \varepsilon.$$

A symmetric argument completes the proof of (3.2). Now, (2.3) and (3.2) give (3.3). Similarly, (2.4) and (3.2) establish (3.4).

To establish the optimality of T, we observe that (3.4) shows $C(T) \leqq 2$. It therefore suffices to show that $C(T') \geqq 2$ for all selection operators T'. To this effect we construct a sequence f_n, $n = 1, 2, ...,$ of functions and a quasi-convex function h, all defined and bounded on $S = [0, 3]$, such that

$$\|T'(f_n) - T'(h)\| / \|f_n - h\| \to 2$$

as $n \to \infty$ for all T'. Indeed, let

$$\begin{aligned} f_n(s) &= 1 - 2s, & 0 &\leqq s < 1, \\ &= -3 + 2s, & 1 &\leqq s < 2, \\ &= 5 + 4/n - 2(1 + 1/n)\, s, & 2 &\leqq s \leqq 3. \\ h(s) &= -2s, & 0 &\leqq s < 1, \\ &= -4 + 2s, & 1 &\leqq s < 2, \\ &= 0, & 2 &\leqq s \leqq 3. \end{aligned}$$

Clearly,

$$\begin{aligned} f'_n(s) &= 2 - 2s, & 0 &\leqq s < 1, \\ &= 0, & 1 &\leqq s < 3 - 1/(n+1) \\ &= 2(3 + 2/n) - 2(1 + 1/n)\, s, & 3 - 1/(n+1) &\leqq s \leqq 3. \end{aligned}$$

This example appears in [11]. It is easy to verify that $T'(f_n)(s) = 0$ for $1 \leqq s \leqq 2$ for any T'. Also, $T'(h) = h$ since h is quasi-convex. Hence,

$$\|T'(f_n) - T'(h)\| \geqq |T'(f_n)(1) - T'(h)(1)| = 2$$

and $\|f_n - h\| = 1 + 2/n$. It follows that $C(T') \geqq 2$. Hence $C(T) = 2$. By (2.1) we have $D \leqq 1$. Now since $\Delta(f_n) = 1$ and $\Delta(h) = 0$, we find that $|\Delta(f_n) - \Delta(h)| / \|f_n - h\| \to 1$. Thus $D = 1$. The proof is complete.

We remark that (2.1) does not imply $D = 1$ for approximation problems in general. This is seen from the example of generalized isotone optimization in [9].

4. Approximation by Convex Functions

In this section, we apply the results of Section 2 to the convex problem. We state our main result.

THEOREM 4.1. *Let $f \in B$ and let $\bar{f}$ be the greatest convex minorant of f. Then $\Delta(f) = (1/2)\| f - \bar{f} \|$, and $f' = \bar{f} + \Delta(f)$ is the maximal best convex approximation to f. Furthermore, if f, $h \in B$, then*

$$\|\bar{f} - \bar{h}\| \leqq \| f - h\|. \tag{4.1}$$

Hence,

$$\|f' - h'\| \leqq \|f - h\|, \qquad \text{if} \quad \Delta(f) = \Delta(h), \tag{4.2}$$

and

$$\|f' - h'\| \leqq 2 \, \|f - h\|. \tag{4.3}$$

The operator $T: B \to K$ defined by $T(f) = f'$ is an optimal Lipschitzian selection operator with $C(T) = 2$. Also $D = 1$.

For the purpose of analysis, we introduce some definitions and terminology. Define a function d' on $(S \times R) \times (S \times R)$ as follows. Let $x = (s, u)$, $y = (t, v)$, where s, $t \in S$, u, $v \in R$, and

$$\begin{aligned} d'(x, y) &= |u - v|, \qquad && \text{if} \quad s = t, \\ &= \infty, && \text{otherwise}. \end{aligned}$$

For $U \subset S \times R$, $x \in S \times R$, and $r > 0$, let

$$d''(x, U) = \inf\{d'(x, y): y \in U\}$$

and

$$B_r(U) = \{x \in S \times R: d''(x, U) < r\}.$$

Analogous to the Hausdorff metric [2], define a function d on the subsets of $S \times R$ by

$$d(U, V) = \inf\{r: U \subset B_r(V) \text{ and } V \subset B_r(U)\},$$

where U, $V \subset S \times R$. Note that $0 \leqq d \leqq \infty$. For any f in B, let $E(f)$ denote the epigraph of f [6, 7], viz.,

$$E(f) = \{(s, u): s \in S, u \in R, u \geqq f(s)\} \subset S \times R.$$

It is easy to verify that if $f, h \in B$, then

$$d(E(f), E(h)) = \|f - h\|. \tag{4.4}$$

For $U \subset S \times R$, we denote by $\mathrm{co}(U)$ the convex hull of U; since $S \times R$ is convex, it is contained in $S \times R$. It is easy to see that for f in B,

$$\bar{f}(s) = \inf\{u\colon (s, u) \in \mathrm{co}(E(f))\}, \qquad s \in S.$$

Consequently, if $f, h \in B$ then

$$d(\mathrm{co}(E(f)), \mathrm{co}(E(h))) = \|\bar{f} - \bar{h}\|. \tag{4.5}$$

Proof of Theorem 4.1. The assertions concerning $\Delta(f)$ and f' follow from Proposition 2.1. To show (4.1), by (4.4) and (4.5) it suffices to show that

$$d(\mathrm{co}(E(f)), \mathrm{co}(E(h))) \leqq d(E(f), E(h)). \tag{4.6}$$

Let $U = E(f)$ and $V = E(h)$. If $r > 0$ and $U \subset B_r(V)$ and $V \subset B_r(U)$, then

$$\mathrm{co}(U) \subset \mathrm{co}(B_r(V)) = B_r(\mathrm{co}(V)).$$

Similarly $\mathrm{co}(V) \subset B_r(\mathrm{co}(U))$. Hence, from the definition of d, (4.6) follows.

We show that T is optimal, exactly as in the proof of Theorem 3.2, by using the following sequence of bounded functions f_n, $n = 1, 2, \ldots$, on $S = [0, 1]$. This sequence appears in [9],

$$\begin{aligned} f_n(s) &= -1 + 2ns, \qquad && 0 \leqq s < 1/n, \\ &= 1, && 1/n \leqq s \leqq 1. \end{aligned}$$

Then $f'_n(s) = 2s - 1/n$, $0 \leqq s \leqq 1$. Let $h = 0$ on $[0, 1]$. It is easy to see that f'_n is the only best approximation to f_n. Consequently, $T'(f_n) = f'_n$ for any selection operator T'. Also $T'(h) = h' = h$. Clearly, $\|f_n - h\| = 1$, $\|f'_n - h'\| = 2 - 1/n$. We therefore have

$$\|T'(f_n) - T'(h)\| / \|f_n - h\| \to 2,$$

as $n \to \infty$. Hence $C(T') \geqq 2$ and thus $C(T) = 2$ by (4.3). Now $\Delta(f_n) = 1 - 1/n$ and $\Delta(h) = 0$. Consequently $D = 1$. The proof is complete.

We now derive a property of $\bar{f}$. We note that the face of the convex set S is convex and is closed relative to S. An extreme point of S is a zero-dimensional face of S. Also S is the disjoint union of relative interiors of faces of S [6].

PROPOSITION 4.1. *Let P be a nonempty face of S. Let $f \in B$ and g be the restriction of f to P. Let $\bar{g}$ be the greatest convex minorant of g (over P). Then $\bar{g}(s) = \bar{f}(s)$ for all s in P. In particular, if s is an extreme point of S, then $f(s) = \bar{f}(s)$.*

Proof. Since the restriction of $\bar{f}$ to P is convex, we have $\bar{g}(s) \geqq \bar{f}(s)$ for all s in P. Now define a function h in B by

$$\begin{aligned} h(s) &= \bar{g}(s), \qquad s \in P, \\ &= \bar{f}(s), \qquad s \in S - P. \end{aligned}$$

We show that h is convex by verifying (1.2) for h. Let $s, t \in S$, $0 < \lambda < 1$, and $u = \lambda s + (1-\lambda)t$. Since $h = \bar{g} \geqq \bar{f}$ on P and P is convex, it suffices to verify (1.2) when u is in P. But since P is a face, this implies that $s, t \in P$. We have $h = \bar{g}$ on P and $\bar{g}$ is convex, hence (1.2) holds for h. Since $\bar{f}$ is the GQM of f, we conclude that $h \leqq \bar{f}$ which gives $\bar{g} \leqq \bar{f}$ on P. The last statement follows from the fact that if s is an extreme point of S, then $\{s\}$ is a face of S. The proof is complete.

5. Two Approximation Problems

Let $f \in B$ and let L_f, $\bar{\Delta}(f)$, and $\bar{D}$ be as defined in Section 1. We consider the two problems of finding a best approximation f' to f from L_f so that $\bar{\Delta}(f) = \|f - f'\|$, when K is the set of quasi-convex (convex) functions. As before, a selection operator T maps an f in B to one of its best approximations f'.

THEOREM 5.1. *Let $f \in B$ and let $\bar{f}$ be the greatest quasi-convex (convex) minorant of f. Then $\bar{f}$ in L_f is the maximal best approximation to f and $\bar{\Delta}(f) = \|f - \bar{f}\|$. The operator $T: B \to K$ defined by $T(f) = \bar{f}$ is an optimal Lipschitzian selection operator with $C(T) = 1$. Also $\bar{D} = 2$.*

Proof. The assertions concerning $\bar{f}$ and $\bar{\Delta}(f)$ follow immediately. Clearly, (3.2) and (4.1) show that $C(T) \leqq 1$. If f and h are two distinct functions in K, then, for any selection operator T', we have $T'(f) = f$ and $T'(h) = h$. Consequently, $C(T') \geqq 1$ and thus $C(T) = 1$. Clearly, $\bar{\Delta}(f) = 2\Delta(f)$ and hence $\bar{D} = 2D = 2$.

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